Multi-faced Janus

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Winter 2015

based on JHEP 03(2014)046

ADS system: Simplest case

Consider 3D Einstein scalar system

$$S = \frac{1}{16\pi G}\int d^3x\sqrt{g}\left(R - g^{ab}\partial_a\phi\partial_b\phi + 2\right)$$

From this action the Einstein equation becomes

$$R_{ab} + 2g_{ab} = \partial_a\phi\partial_b\phi$$

and the scalar equation of motion is given by

$$\partial_a(\sqrt{g}g^{ab}\partial_b\phi) = 0$$

With $\phi = 0$, one may find the usual ADS solution in 3D: (in Poincare coord.)

$$ds_3^2 = \frac{-dt^2 + dx^2 + d\xi^2}{\xi^2}$$

At the boundary ($\xi \to 0$), the space-time is just $ds_2^2 = -dt^2 + dx^2$.

Janus solution: Now with $\phi \neq 0$

We take an Ansatz:

$$ds_3^2 = dr^2 + f(r)\frac{-dt^2 + d\xi^2}{\xi^2}, \qquad \phi = \phi(r)$$

The solutions of Einstein equation and matter equation: (Bak, Gutperle and Hirano, 2007)
the metric sector:

$$f(r) = \frac{1}{2}\left(1 + \sqrt{1 - 2\gamma^2}\cosh(2r)\right)$$

the dilaton sector:

$$\phi(r) = \frac{1}{\sqrt{2}}\log\left(\frac{1 + \sqrt{1 - 2\gamma^2} + \sqrt{2}\gamma\tanh(r)}{1 + \sqrt{1 - 2\gamma^2} - \sqrt{2}\gamma\tanh(r)}\right)$$

Note that the dilaton approaches two constant values at the boundaries

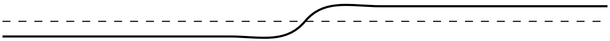
$$\lim_{r \to \pm\infty}\phi(r) = \frac{1}{\sqrt{2}}\log\left(\frac{1 + \sqrt{1 - 2\gamma^2} \pm \sqrt{2}\gamma}{1 + \sqrt{1 - 2\gamma^2} \mp \sqrt{2}\gamma}\right) \equiv \pm\phi_{as}$$

For later use it is also convenient to express γ in terms ϕ_{as}:

$$\gamma = \frac{1}{\sqrt{2}} \tanh \sqrt{2}\phi_{as}$$

For small value of γ, dilaton field has the expansion

$$\phi(r) = \gamma \tanh r + \frac{\gamma^3}{6}(3 \tanh r + \tanh^3 r) + \cdots$$



Janus deformation in Poincare coordinates

Introducing an angular coordinate μ by

$$dr = \sqrt{f(r)}d\mu\,,$$

this solution can be presented by

$$ds_3^2 = f(\mu)\left(d\mu^2 + \frac{-dt^2 + d\xi^2}{\xi^2}\right), \qquad \phi = \phi(\mu)$$

where

$$\begin{aligned} f(\mu) &= \frac{\kappa_+^2}{\mathrm{sn}^2(\kappa_+(\mu+\mu_0), k^2)} = \frac{\kappa_+^2\,\mathrm{dn}^2(\kappa_+\mu, k^2)}{\mathrm{cn}^2(\kappa_+\mu, k^2)} \\ \phi(\mu) &= \sqrt{2}\ln\left(\mathrm{dn}(\kappa_+(\mu+\mu_0), k^2) - k\,\mathrm{cn}(\kappa_+(\mu+\mu_0), k^2)\right) \\ &= \frac{1}{\sqrt{2}}\ln\left(\frac{1 - k\,\mathrm{sn}(\kappa_+\mu, k^2)}{1 + k\,\mathrm{sn}(\kappa_+\mu, k^2)}\right) \end{aligned}$$

with

$$\kappa_\pm^2 \equiv \frac{1}{2}(1 \pm \sqrt{1-2\gamma^2}), \quad k^2 \equiv \kappa_-^2/\kappa_+^2 = \frac{\gamma^2}{2} + O(\gamma^4)$$

$$\mu_0 \equiv K(k^2)/\kappa_+ = \frac{\pi}{2}\left(1 + \frac{3}{8}\gamma^2 + O(\gamma^4)\right)$$

This describes Janus deformation of the Poincare patch geometry.

Planar BTZ Black Holes

Three dimensional Janus black holes corresponds to a dilaton deformation of the planar BTZ black hole solution.
The Eulclidean BTZ black hole in 3D (Banados:1992) can be written as

$$ds^2 = \frac{1}{z^2}\left[(1-z^2)d\tau^2 + dx^2 + \frac{dz^2}{1-z^2}\right]$$

$z = 1$: the horizon $\quad z = 0$: ADS boundary.
It is regular at $z = 1$ if the Euclidean time τ has a periodicity 2π. To see this, let us change the variable

$$\tilde{z}^2 = 1 - z^2$$

In this coordinate system the black hole looks like

$$ds^2 = \frac{1}{1-\tilde{z}^2}\left[\tilde{z}^2 d\tau^2 + dx^2 + \frac{d\tilde{z}^2}{1-\tilde{z}^2}\right],$$

and now the AdS boundary is located at $\tilde{z} = 1$ while the horizon at $\tilde{z} = 0$.
Note

$$\frac{d\tilde{z}^2}{1-\tilde{z}^2} + \tilde{z}^2 d\tau^2 \sim d\tilde{z}^2 + \tilde{z}^2 d\tau^2$$

Therefore the corresponding temperature can be identified as

$$T = \frac{1}{2\pi}$$

The BTZ black hole with general temperature is described by the metric

$$ds^2 = \frac{1}{z'^2}\left[(1 - a^2 z'^2)d\tau'^2 + dx'^2 + \frac{dz'^2}{1 - a^2 z'^2}\right]$$

which can be obtained by the scale coordinate transformation

$$z' = az \qquad \tau' = a\tau \qquad x' = ax$$

from (6). The temperature for this scaled version of the black hole now becomes

$$T' = \frac{a}{2\pi}$$

Below we work mostly with the temperature $T = (2\pi)^{-1}$

Ansatz for the Black Janus

With $z = \sin y$, the BTZ black hole can be rewritten as

$$ds^2 = \frac{1}{\sin^2 y}\left[\cos^2 y \, d\tau^2 + dx^2 + dy^2\right]$$

Motivated by the form of this metric, we shall make the following ansatz for the black Janus solution

$$ds^2 = \frac{dx^2 + dy^2}{A(x,y)} + \frac{d\tau^2}{B(x,y)}, \qquad \phi = \phi(x,y)$$

It is then straightforward to show that the equations of motion reduce to

$$(\vec{\partial} A)^2 - A\,\vec{\partial}^2 A = 2A - A^2\,(\vec{\partial}\phi)^2$$
$$3(\vec{\partial} B)^2 - 2B\,\vec{\partial}^2 B = 8B^2/A$$
$$\vec{\partial} B \cdot \vec{\partial}\phi - 2B\,\vec{\partial}^2\phi = 0$$

where $\vec{\partial} = (\partial_x, \partial_y)$.

Perturbative approach

As a power series in γ, the scalar field may be expanded as

$$\phi(x, y) = \gamma\phi_0(x, y) + \gamma^3\phi_3(x, y) + O(\gamma^5)$$

Then the scalar equation in the leading order becomes

$$\tan y\, \partial_y \varphi - \sin^2 y\, \vec{\partial}^2 \varphi = 0$$

where $\varphi(x, y)$ denotes $\phi_0(x, y)$
The leading perturbation of the metric part is of order γ^2. For which we set

$$A = A_0\Big(1 + \frac{\gamma^2}{4} a(x, y) + O(\gamma^4)\Big), \qquad B = B_0\Big(1 + \frac{\gamma^2}{4} b(x, y) + O(\gamma^4)\Big)$$

with

$$A_0 = \sin^2 y, \qquad B_0 = \tan^2 y$$

The leading order equations for the metric part becomes

$$2a - \sin^2 y\, \vec{\partial}^2 a = -4 \sin^2 y (\vec{\partial}\varphi)^2$$

$$2 \tan y\, \partial_y b - \sin^2 y\, \vec{\partial}^2 b + 4a = 0$$

Linearized Black Janus

Using the Janus boundary condition $\phi(x,0) = \gamma\,\epsilon(x) + O(\gamma^3)$ with the sign function $\epsilon(x)$, the leading order scalar equation is solved by

$$\varphi = \frac{\sinh x}{\sqrt{\sinh^2 x + \sin^2 y}}$$

The solution for the geometry part can be found as $a(x,y) = b(x,y) = q(x,y)$ where

$$q(x,y) = 3\left(\frac{\sinh x}{\sin y}\right)\tan^{-1}\left(\frac{\sinh x}{\sin y}\right) + \frac{\sinh^2 x}{\sinh^2 x + \sin^2 y} + 2 + 2C\frac{\sinh x}{\sin y}$$

with an integration constant C.
Then the metric for the black Janus can be written as

$$ds^2 = \frac{1 - \frac{\gamma^2}{4}q(x,y)}{\sin^2 y}\left[\cos^2 y\, d\tau^2 + dx^2 + dy^2\right] + O(\gamma^4)$$

$\longrightarrow$**Exact solution!!**

Exact solution for Black Janus

D. Bak, M. Gutperle and R. A. Janik, JHEP **1110**, 056 (2011)

$$ds^2 = \cot^2 u d\tau^2 + F(u,\varphi)\Big[\frac{du^2}{\sin^2 u} + \cot u\,(\log f(\varphi))' du d\varphi$$
$$+\frac{\phi_{as}^2 f^2(\varphi)}{\gamma^4}\Big(\gamma^2 \sin^2 u + \cos^2 u\Big[1 - \frac{\cosh\sqrt{2}\phi_{as}\varphi}{\cosh\sqrt{2}\phi_{as}} + \frac{\sinh^2\sqrt{2}\phi_{as}\varphi}{2\cosh^2\sqrt{2}\phi_{as}}\Big]\Big)d\varphi^2\Big]$$

where

$$F(u,\varphi) = \Big[\sin^2 u + \frac{\cos^2 u}{f(\varphi)}\Big]^{-1}$$

and

$$f(\varphi) = \frac{\gamma^2}{1 - \frac{\cosh\sqrt{2}\phi_{as}\varphi}{\cosh\sqrt{2}\phi_{as}}}$$

Scalar field at boundary

We begin with the scalar field perturbation.

$$\left[\partial_x^2 + 4s\partial_s\,(1-s)\,\partial_s\right]\varphi(x,s) = 0$$

where we introduce $s = \sin^2 y$.
This is solved by

$$\varphi(x,s) = \int dk\tilde{\varphi}(k)e^{ikx}F(ik/2,-ik/2;1;1-s)|\Gamma(1+ik/2)|^2$$

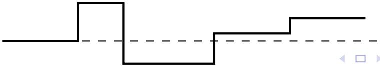
where $F(a,b;c;x)$ is the hypergeometric function

$$F(a,b;c;z) = \sum_{n=0}^{\infty}\frac{(a)_n(b)_n}{(c)_n}z^n$$

the scalar perturbation at the boundary $y = 0$ (or $s = 0$):

$$\varphi(x,s) = \varphi_0(x) + s\varphi_1(x) + s^2\varphi_2(x) + \cdots$$

The regularity of field equation near $s = 0$ shows that $\varphi_0(x)$ has to be piecewise constant



Example I, Black Janus

First we consider Black Janus: $\varphi(x, s = 0) = \varphi_0(x) = \epsilon(x)$.
For this case, one finds $\tilde{\varphi}(k) = 1/(\pi i k)$ leading to

$$\varphi(x, s) = \int_0^\infty dk \frac{2 \sin kx}{\pi k} F(ik/2, -ik/2; 1; 1 - s) |\Gamma(1 + ik/2)|^2$$

Noting $F(ik/2, -ik/2; 1; 0) = 1$ and $|\Gamma(1 + ik/2)|^2 = \frac{\pi k/2}{\sinh \pi k/2}$, one finds

$$\varphi(x, s = 1) = \int_0^\infty dk \frac{\sin kx}{\sinh \pi k/2} = \tanh x$$

Then using this, one has

$$\begin{aligned} \varphi(x, s) &= \left[1 + (1 - s) \frac{-\frac{1}{4}\partial_x^2}{1!^2} + (1 - s)^2 \frac{\left(-\frac{1}{4}\partial_x^2\right)\left(1 - \frac{1}{4}\partial_x^2\right)}{2!^2} + \cdots \right] \tanh x \\ &= \frac{\sinh x}{\sqrt{\sinh^2 x + s}} \end{aligned} \tag{1}$$

Example II, Black Janus with three faces

The second is the example where $\varphi(x, s = 0)$ is given by

$$\varphi(x, s = 0) = \left[\begin{array}{ll} 2 & \text{for } 0 \leq x \leq l \\ 0 & \text{otherwise} \end{array} \right.$$

For this case,

$$\varphi(x, s) = \int_0^\infty dk \frac{\cos k(x - l/2) \sin kl}{\sinh \pi k/2} F(ik/2, -ik/2; 1; 1 - s)$$

leading to

$$\varphi(x, s = 1) = \frac{2 \sinh l}{\cosh(2x - l) + \cosh l} = \tanh x - \tanh(x - l)$$

And

$$\varphi(x, s) = \varphi_0(x, s) - \varphi_0(x - l, s)$$

with $\varphi_0(x, s) = \frac{\sinh x}{\sqrt{\sinh^2 x + s}}$

Example III, periodic case

Consider the periodic case with a period $2l$:

$$\varphi(x, s=0) = \left[\begin{array}{ll} 1 & \text{for} \quad 0 \le x < l \\ -1 & \text{for} \ -l \le x < 0 \end{array} \right.$$

With $k_m = \pi(2m+1)/l$, one finds

$$\varphi(x, s) = \frac{2\pi}{l} \sum_{m=0}^{\infty} \frac{\sin k_m x}{\pi k_m/2} F(ik_m/2, -ik_m/2; 1; 1-s) |\Gamma(1 + ik_m/2)|^2$$

This sum can be written in terms of elliptic functions.
For a further analysis, it is convenient to use an alternative expression. Using the translational symmetry and linearity, We may construct a periodic solution in the form:

$$\varphi(x, s) = \sum_{n=-\infty}^{\infty} (-1)^n \varphi_0(x - nl, s)$$

using the solution $\varphi_0(x, s) = \frac{\sinh(x)}{\sqrt{\sinh^2(x)+s}}$ again.

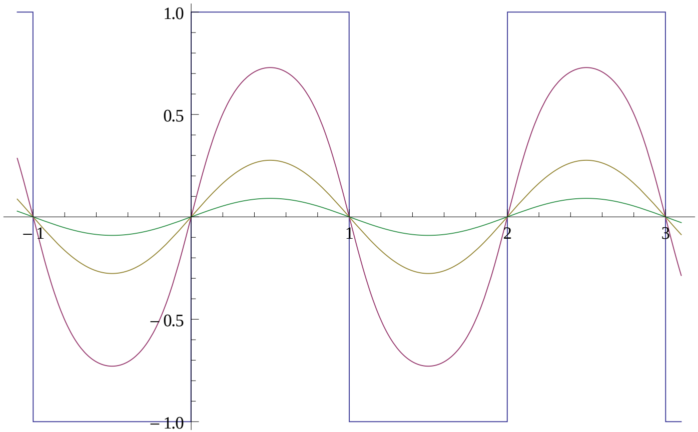


Figure: Some plots of the function (2) for $s = 0, 1/10, 1/2, 1$ (from the above)

Generic case

In generic, if the $\varphi(x,0)$ is give as, at the boundary,

$$\varphi(x,0) = \sum_{n=-\infty}^{\infty} \alpha_n \epsilon(x - l_n)$$

we have the solution

$$\varphi(x,s) = \sum_{n=-\infty}^{\infty} \alpha_n \varphi_0(x - l_n, s)$$

with $\varphi_0(x,s) = \frac{\sinh x}{\sqrt{\sinh^2 x + s}}$

Gravity sector

Remember the Einstein eq.

$$(\vec{\partial}A)^2 - A\,\vec{\partial}^2 A = 2A - A^2\,(\vec{\partial}\phi)^2$$
$$3(\vec{\partial}B)^2 - 2B\,\vec{\partial}^2 B = 8B^2/A$$
$$\vec{\partial}B\cdot\vec{\partial}\phi - 2B\,\vec{\partial}^2\phi = 0$$

The back reaction: (in the first leading order of γ

$$4\sin^2 y(\vec{\partial}\varphi)^2 = 4\sin^2 y \sum_{n_1,n_2} \alpha_{n_1}\alpha_{n_2}\vec{\partial}\varphi_0(x-l_{n_1},y)\cdot\vec{\partial}\varphi_0(x-l_{n_2},y)$$

Split this into the "diagonal" part

$$4\sin^2 y \sum_{n=-\infty}^{\infty} \alpha_n^2(\vec{\partial}\varphi_0(x-l_n,y))^2$$

and the "off-diagonal" part

$$4\sin^2 y \sum_{n_1<n_2} 2\alpha_{n_1}\alpha_{n_2}\vec{\partial}\varphi_0(x-l_{n_1},y)\cdot\vec{\partial}\varphi_0(x-l_{n_2},y).$$

Diagonal parts

Each source term for the diagonal part is the same as that for the black Janus up to the translation in x. One may then easily find the solution for the diagonal part

$$a^{\text{diag}}(x,y) = b^{\text{diag}}(x,y) = \sum_{n=-\infty}^{\infty} \alpha_n^2 \left[q_0(x-l_n, y) + C_n \frac{\sinh(x-l_n)}{\sin y} \right],$$

where

$$q_0(x,y) = 3\,\frac{\sinh x}{\sin y} \left(\tan^{-1}\left(\frac{\sinh x}{\sin y}\right) - \frac{\pi}{2} \right) + 2 + \frac{\sin^2 y}{\sinh^2 x + \sin^2 y}.$$

Later we shall show that the different choice of C_n are all related by an appropriate coordinate transformation. Hence we can set $C_n = 0$ without loss of generality.

Off-diagonal part

Because of translational invariance in x direction, it suffices to consider the case of $n_1 = 0$ and $n_2 = 1$ with $l_0 = 0$ and $l_1 = l$. This leads to the equations

$$-2a_c(x,y,l) + \sin^2 y\, \vec{\partial}^2 a_c(x,y,l) = \frac{4(\cosh l + XY)}{(1+X^2)^{3/2}(1+Y^2)^{3/2}},$$

$$-2\tan y\, \partial_y b_c(x,y,l) + \sin^2 y\, \vec{\partial}^2 b_c(x,y,l) = 4a_c(x,y;l),$$

where we have introduced

$$X = \frac{\sinh x}{\sin y}, \quad Y = \frac{\sinh(x-l)}{\sin y}.$$

Once the solution of the above equations is given, the full off-diaginal part of the solution may be given in the form

$$a^{\rm off}(x,y) = \sum_{n_1<n_2} 2\alpha_{n_1}\alpha_{n_2}\, a_c(x - l_{n_1}, y, l_{n_2} - l_{n_1}),$$

$$b^{\rm off}(x,y) = \sum_{n_1<n_2} 2\alpha_{n_1}\alpha_{n_2}\, b_c(x - l_{n_1}, y, l_{n_2} - l_{n_1}).$$

We take the range of x and y as $0 < x < l$ and $0 \le \sin y \le 1$. Then X and Y can take positive real values.

Remarkably, it turns out that we can find a solution of this equation by taking the ansatz

$$a_c(x, y, l) = \frac{1}{\sqrt{1+X^2}\sqrt{1+Y^2}} G(m), \quad b_c(x, y, l) = H(m)$$

where

$$m = (\sqrt{1+X^2} - X)(\sqrt{1+Y^2} - Y).$$

After some works, we have found the solution in the form

$$G(m) = \mathcal{G}(m, \cosh l) \equiv 2m - \mathcal{I}(m, \cosh l) G_h(m, \cosh l),$$

where the integral $\mathcal{I}(x, \cosh l)$ is given by

$$\mathcal{I}(m, \cosh l) = \int_0^m dx \left(\frac{x}{1 + x^2 + 2x \cosh l} \right)^{\frac{3}{2}}.$$

with the homogeneous solution to this equation,

$$G_h(m, \cosh l) = \frac{1}{m^{3/2}} (1 - m^2) \sqrt{1 + m^2 + 2m \cosh l},$$

Off diagonal parts: final form

Including the homogeneous part, we have

$$a_c(x, y, l) = \frac{1}{\sqrt{1+X^2}\sqrt{1+Y^2}} \Big(\mathcal{G}(m, \cosh l) + C_G G_h(m, \cosh l)\Big),$$
$$b_c(x, y, l) = \frac{4m}{1+m^2+2m\cosh l} \Big(\mathcal{G}(m, \cosh l) + C_G G_h(m, \cosh l)\Big).$$

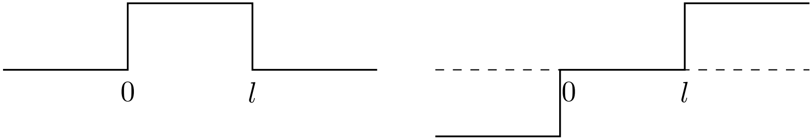
Remember

$$a^{\text{off}}(x, y) = \sum_{n_1<n_2} 2\alpha_{n_1}\alpha_{n_2}\, a_c(x - l_{n_1}, y, l_{n_2} - l_{n_1}),$$
$$b^{\text{off}}(x, y) = \sum_{n_1<n_2} 2\alpha_{n_1}\alpha_{n_2}\, b_c(x - l_{n_1}, y, l_{n_2} - l_{n_1}).$$

Consider the boundary condition for the scalar given by

$$\varphi(x,0) = \alpha_- \epsilon(x) + \alpha_+ \epsilon(x-l) \to \epsilon(x) - \epsilon(x-l).$$

(with $\alpha_- = \alpha_+ = 1$.)



(a) (b)

Figure: (a) describes the boundary condition, $\varphi(x,0)$, for $\alpha_- = -\alpha_+ = 1$ while (b) for $\alpha_- = \alpha_+ = 1$.

The full geometric part of the solution takes the form

$$a(x, y, l) = q_0(x, y) + q_0(x - l, y) + 3\pi Y - 2a_c^0(x, y, l),$$
$$b(x, y, l) = q_0(x, y) + q_0(x - l, y) + 3\pi Y - 2b_c^0(x, y, l),$$

where $a_c^0(x, y, l)$ and $b_c^0(x, y, l)$ represent the unit-coefficient cross-term solution given by

$$a_c^0(x, y, l) = \frac{1}{\sqrt{1 + X^2}\sqrt{1 + Y^2}} \left(\mathcal{G}(m, \cosh l) + \frac{1}{2}\Delta(l) G_h(m, \cosh l)\right),$$
$$b_c^0(x, y, l) = \frac{4m}{1 + m^2 + 2m\cosh l} \left(\mathcal{G}(m, \cosh l) + \frac{1}{2}\Delta(l) G_h(m, \cosh l)\right).$$

with

$$\Delta(l) \equiv \mathcal{I}(\infty, \cosh l) = \frac{e^{\frac{l}{2}}}{\sinh^2 l} \left(\cosh l \; E(1 - e^{-2l}) - e^{-l} K(1 - e^{-2l})\right).$$

Shape of the boundary

For the region $0 < x < l$ there is no singular term in $q_0(x, y)$ and $q_0(x - l, y) + 3\pi Y$. They behave

$$q_0(x, y) = -\frac{2\sin^4 y}{5\sinh^4 x} + O(\sin^6 y), \qquad \text{for} \quad 0 \le x,$$
$$q_0(x - l, y) + 3\pi Y = -\frac{2\sin^4 y}{5\sinh^4 (x - l)} + O(\sin^6 y), \qquad \text{for} \quad x \le l.$$

On the other hand, the cross term behaves

$$a_c(x, y) = a_1^c(x)\sin^2 y + a_2^c(x)\sin^4 y + O(\sin^6 y),$$
$$b_c(x, y) = b_0^c(x) + b_1^c(x)\sin^2 y + O(\sin^4 y)$$

for $0 < x < l$.

Shape of the boundary

Compare the results with the original BTZ metric without any deformation.

$$ds^2 = \frac{1}{\sin^2 y}\left[\frac{dx^2 + dy^2}{1 + \frac{\gamma^2}{4} a_h(x,y) + O(\gamma^4)} + \frac{\cos^2 y \; d\tau^2}{1 + \frac{\gamma^2}{4} b_h(x,y) + O(\gamma^4)}\right]$$

where a_h and b_h denote the homogeneous part of the solution. Then by the coordinate transformation,

$$y' = y + \frac{\gamma^2}{4}\cos y \; \mathcal{A}_h(x,y) + O(\gamma^4),$$
$$x' = x + \frac{\gamma^2}{4}\cosh x \, \mathcal{B}_h(x,y) + O(\gamma^4),$$

one can bring the metric into the original BTZ form

$$ds^2 = \frac{1}{\sin^2 y'}\left[dx'^2 + dy'^2 + \cos^2 y' \; d\tau^2\right].$$

The resulting form of the coordinate transformation for y' reads explicitly

$$y' = y + \frac{\gamma^2}{8} \left[3\pi \sinh(x-l)\cos y - 4\Delta(l) \frac{\frac{1}{m} - m}{\sqrt{m + \frac{1}{m} + 2\cosh l}} \sin y \cos y \right] + O$$

for $x \geq l$. The boundary defined by $y' = 0$ is determined by the equation

$$\sin y = \frac{\gamma^2}{8} \left[4\Delta(l) \frac{\frac{1}{m} - m}{\sqrt{m + \frac{1}{m} + 2\cosh l}} \sin y - 3\pi \sinh(x-l) \right] + O(\gamma^4).$$

The solution has the form

$$\sin y = \gamma^2 f(x) + O(\gamma^4)$$

with some function $f(x)$.

Boundaries

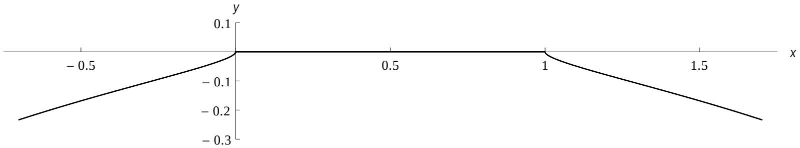


Figure: The shape of the boundary in (x, y) space is depicted for $\gamma^2 = 0.1$, $l = 1$ and $\alpha_- = \alpha_+ = 1$.

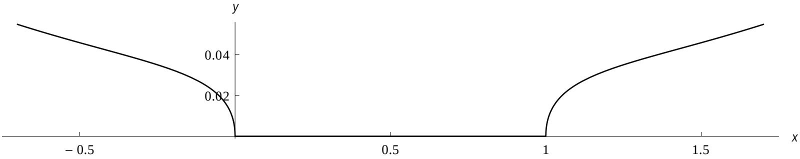


Figure: The shape of the boundary in (x, y) space is depicted for $\gamma^2 = 0.1$, $l = 1$ and $\alpha_- = -\alpha_+ = 1$.

Remarks

- We have developed a way to find ADS solution with dilaton which has piece-wise constant values at the boundary, at least perturbatively.
- For the case with double interfaces (or three faces Janus), explicit forms of the solutions are given. They are expressed in terms of elliptic integrals.
- Entanglement entropy and Casimir energy have been calculated based on this solution.
- The case of lattice is quite interesting.
- Numerical study is under investigation.